MATH 2040 Lecture 13 (24/10/2016)

§ Gram-Schmidt Process

Recall: $S = \{\vec v_1, \dots, \vec v_k\}$ orthogonal, $\vec v_i \neq \vec 0$ for all i $\Longrightarrow$ S lin. indep.

$\overset{?}{\Longleftarrow}$

Gram-Schmidt orthogonalization process

E.g. ($\mathbb{R}^2$) $S = \left\{ \binom{1}{0}, \binom{1}{1} \right\}$ lin. indep.

$$S = \{\vec w_1, \vec w_2\} \xrightarrow{GS} S' = \{ \underset{\overset{\shortparallel}{\vec v_1}}{\vec w_1}, \underbrace{\vec w_2 - c\vec w_1}_{\overset{\shortparallel}{\vec v_2}} \}$$

lin. indep. $\qquad \vec v_1 \perp \vec v_2$

Note: (1) Span S = Span S'

(2) $\vec v_1 \perp \vec v_2 \iff \langle \vec w_1, \vec w_2 - c\vec w_1 \rangle = 0$

$$\iff \langle \vec w_1, \vec w_2 \rangle - \bar{c} \langle \vec w_1, \vec w_1 \rangle = 0$$

$$\iff \boxed{c = \frac{\langle \vec w_2, \vec w_1 \rangle}{\langle \vec w_1, \vec w_1 \rangle}}$$

$$\left.\begin{aligned}\vec{v}_1 &= \begin{pmatrix}1\\0\end{pmatrix}\\ \vec{v}_2 &= \begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{1^2}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}\end{aligned}\right\rangle \text{orthogonal.}$$

E.g. (3 vectors)

$(\mathbb{R}^3)$ $S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

$$\vec{w}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \vec{w}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Idea: Make them $\perp$ "one vector at a time."

$$\vec{v}_1 = \vec{w}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{w}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{3}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}2/5\\-1/5\\1\end{pmatrix} \perp \vec{v}_1.$$

$$\vec{v}_3 = \vec{w}_3 - \underbrace{\frac{\langle \vec{w}_3, \vec{v}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1}_{\perp \vec{v}_1} - \underbrace{\frac{\langle \vec{w}_3, \vec{v}_2\rangle}{\langle \vec{v}_2, \vec{v}_2\rangle}\vec{v}_2}_{\perp \vec{v}_2}$$

$$\|\cdot\| = \frac{1}{5}(2^2+1^2+5^2)^{1/2} = \frac{\sqrt{30}}{5}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} - \frac{2/5}{\sqrt{30}/5}\begin{pmatrix}2/5\\-1/5\\1\end{pmatrix} = \cdots\cdots$$

Note: $\operatorname{span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{span}\{\vec{w}_1, \vec{w}_2\}$

$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

Span remains the same after each step.

Thm: (Gram-Schmidt Process)

Given a lin. indep. $S = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$ in $(V, \langle , \rangle)$,

$\Rightarrow \exists$ orthogonal subset $S' = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$

s.t. $\text{Span}\{\underbrace{\vec{w}_1, \dots, \vec{w}_i}_{\subseteq S}\} = \text{Span}\{\underbrace{\vec{v}_1, \dots, \vec{v}_i}_{\subseteq S'}\}$ for ALL i.

More explicitly,

$$\text{GS}\begin{cases} \vec{v}_1 = \vec{w}_1 \\ \vec{v}_2 = \vec{w}_2 - \dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 \\ \vdots \qquad\quad \vdots \\ \vec{v}_k = \vec{w}_k - \displaystyle\sum_{j=1}^{k-1} \dfrac{\langle \vec{w}_k, \vec{v}_j \rangle}{\langle \vec{v}_j, \vec{v}_j \rangle} \vec{v}_j \end{cases}$$

To produce O.N.B.:

$$\text{Any ordered basis} \xrightarrow{\text{GS}} \text{orthogonal basis} \xrightarrow{\text{normalize}} \text{O.N.B.}$$

Corollary: Any $(V, \langle , \rangle)$ has an O.N.B. $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$

So, then any $\vec{v} \in V$ can be expressed

$$\vec{v} = \underbrace{\langle \vec{v}, \vec{v}_1 \rangle}\, \vec{v}_1 + \underbrace{\langle \vec{v}, \vec{v}_2 \rangle}\, \vec{v}_2 + \dots + \underbrace{\langle \vec{v}, \vec{v}_n \rangle}\, \vec{v}_n$$

Fourier coefficients

E.g. (Legendre polynomials)

$$V = \underbrace{P_2(\mathbb{R})}_{\substack{\text{poly. of} \\ \deg \leq 2 \\ (\mathbb{R}\text{-coeff.})}} \leq C([-1,1]) \quad , \quad \langle f, g \rangle := \int_{-1}^{1} f(t) g(t)\, dt$$

Start with std basis for $P_2(\mathbb{R})$

$$\beta = \{ \underset{\vec{w}_1}{\overset{}{\underset{\parallel}{1}}} , \underset{\vec{w}_2}{\underset{\parallel}{x}} , \underset{\vec{w}_3}{\underset{\parallel}{x^2}} \}$$

lin. indep.

BuT <u>NOT</u> o.N.B. w.r.t. $\langle , \rangle$

Apply Gram-Schmidt,

$$\vec{v}_1 = \vec{w}_1 = 1$$

Compute inner products,

$$\langle \vec{w}_2 , \vec{v}_1 \rangle = \langle x , 1 \rangle := \int_{-1}^{1} x\, dx = 0$$

$$\langle \vec{v}_1 , \vec{v}_1 \rangle = \langle 1 , 1 \rangle := \int_{-1}^{1} 1\, dx = 2$$

$$\Rightarrow \quad \vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2 , \vec{v}_1 \rangle}{\langle \vec{v}_1 , \vec{v}_1 \rangle} \vec{v}_1 = x - \frac{0}{2} \cdot 1 = x$$

Compute inner products,

$$\langle \vec{w}_3 , \vec{v}_1 \rangle = \langle x^2 , 1 \rangle = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$$

$$\langle \vec{w}_3 , \vec{v}_2 \rangle = \langle x^2 , x \rangle = \int_{-1}^{1} x^3\, dx = 0$$

$$\langle \vec{v}_2 , \vec{v}_2 \rangle = \langle x , x \rangle = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$$

$$\Rightarrow \quad \vec{v}_3 = \vec{w}_3 - \frac{\langle \vec{w}_3 , \vec{v}_1 \rangle}{\langle \vec{v}_1 , \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{w}_3 , \vec{v}_2 \rangle}{\langle \vec{v}_2 , \vec{v}_2 \rangle} \vec{v}_2$$

$$= x^2 - \frac{2/3}{2} \cdot 1 - 0 = x^2 - \frac{1}{3}$$

$\therefore \{1, x, x^2 - \frac{1}{3}\}$ orthogonal basis

$\big\downarrow$ normalize

$$\beta' = \Big\{ \underbrace{\tfrac{1}{\sqrt{2}}, \sqrt{\tfrac{3}{2}}\,x, \sqrt{\tfrac{5}{8}}(3x^2-1)}_{\text{Legendre polynomials}} \Big\} \quad \text{O.N.B.}$$

§ Orthogonal complement

$(V, \langle , \rangle)$: inner product space / $\mathbb{R}$ or $\mathbb{C}$

(dim V could be $+\infty$)

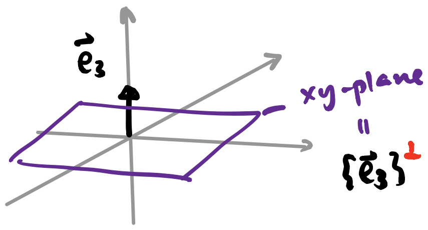
Def$^{\underline{n}}$: Take any subset $\phi \neq S \subseteq V$, define

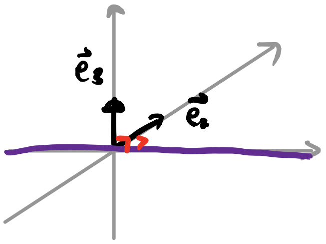
$$S^{\perp} := \{ \vec{x} \in V : \langle \vec{x}, \vec{v} \rangle = 0 \ \ \underline{\underline{\forall}}\ \vec{v} \in S \}$$

orthogonal complement of S

Eg: Trivial: $\{\vec{0}\}^{\perp} = V$, $V^{\perp} = \{\vec{0}\}$

$\underline{\mathbb{R}^3}$: $\{\vec{e}_3\}^{\perp} = $ xy-plane $\qquad \{\vec{e}_2, \vec{e}_3\}^{\perp} = $ x-axis





Remarks: (1) $S^\perp = (\operatorname{span} S)^\perp$ $\quad$ ∴ WLOG, assume S is a subspace

(2) $S^\perp$ is always a subspace! (even when S is not)

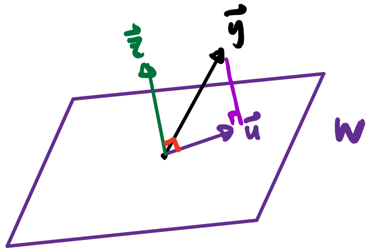
Theorem: If $W \subseteq (V, \langle,\rangle)$ is a subspace, $\dim W < +\infty$.

then $\boxed{V = W \oplus W^\perp}$ $\quad$ (Remark: true for $\dim V = +\infty$)

i.e. any $\vec{y} \in V$ can be uniquely decomposed as

$$\vec{y} = \underset{\substack{\cap \\ W}}{\vec{u}} + \underset{\substack{\cap \\ W^\perp}}{\vec{z}}$$

"orthogonal" projection of $\vec{y}$ onto W



Proof: "Existence": $V = W + W^\perp$

W finite dim $\Rightarrow$ $\beta = \{\vec{v}_1, \dots, \vec{v}_k\}$ O.N.B. for W

For any $\vec{y} \in V$, define

$$\vec{u} := \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i \in W$$

Note: $\langle \vec{u}, \vec{v}_i \rangle = \langle \vec{y}, \vec{v}_i \rangle$

since β O.N.B.

Check: $\vec{z} := \vec{y} - \vec{u} \in W^\perp$ ✓

$$\langle \vec{z}, \underset{\uparrow}{\vec{x}} \rangle = \langle \vec{z}, \sum_{i=1}^{k} a_i \vec{v}_i \rangle = \sum_{i=1}^{k} \overline{a_i} \underbrace{\langle \vec{z}, \vec{v}_i \rangle}_{\substack{\| \\ \langle \vec{y} - \vec{u}, \vec{v}_i \rangle = 0}} = 0$$

any vector in W

"Uniqueness": $W \cap W^{\perp} = \{\vec{0}\}$

Take any $\vec{y} \in W \cap W^{\perp}$,

$$\langle \underset{W}{\underset{\cap}{\vec{y}}}, \underset{W^{\perp}}{\underset{\cap}{\vec{y}}} \rangle = 0 \quad \Rightarrow \quad \vec{y} = \vec{0}.$$

Geometrically, $\vec{y} = \underset{W}{\underset{\cap}{\vec{u}}} + \underset{W^{\perp}}{\underset{\cap}{\vec{z}}}$

then "$\vec{u}$ is the unique vector in W which is closest to $\vec{y}$"

i.e. $\forall \vec{x} \in W$, $\|\vec{y} - \vec{x}\| \geq \|\vec{y} - \vec{u}\|$ —— (*)

$\Rightarrow$ Pythagoras Thm: $\boxed{\|x+y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{if } \langle x, y \rangle = 0.}$

Ex: Prove (*) using ↑ .

Corollary: Assume $\dim V < +\infty$.

(a) $$\underset{\text{O.N. basis}}{\{\vec{v}_1, \dots, \vec{v}_k\}} \subseteq W \xrightarrow{\text{extend}} \overset{\text{O.N.B.}}{\{\underbrace{\vec{v}_1, \dots, \vec{v}_k}_{W}, \underbrace{\vec{v}_{k+1}, \dots, \vec{v}_n}_{W^{\perp}}\}} \subseteq V$$

(b) $\dim V = \dim W + \dim W^{\perp}$